

# A Note on Anomaly Matching for Finite Density QCD

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## Abstract

We note that the QCD phases at large finite density respect 't Hooft anomaly matching conditions. Specifically the spectrum of the light excitations possesses the correct quantum numbers required to obey global anomaly constraints. We argue that 't Hooft constraints can be used at finite density along with non perturbative methods to help selecting the correct phase.

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## I. INTRODUCTION

Recently quark matter at very high density has attracted a great flurry of interest [1–4]. In this regime quark matter is expected to behave as a color superconductor [1,2]. Possible phenomenological applications are associated with the description of neutron star interiors, neutron star collisions and the physics near the core of collapsing stars. A better understanding of highly squeezed nuclear matter might also shed some light on nuclear matter at low density, i.e. densities close to ordinary nuclear matter where some model already exist. See for example Ref. [5] where a rather complete soliton model at low density is constructed containing along with the Goldstone bosons also vector-bosons.

In a superconductive phase, the color symmetry is spontaneously broken and a hierarchy of scales, for given chemical potential, is generated. Indicating with  $g$  the underlying coupling constant the relevant scales are: the chemical potential  $\mu$  itself, the dynamically generated gluon mass  $m_{gluon} \sim g\mu$  and the gap parameter  $\Delta \sim \mu e^{-\frac{1}{g}}$ . Since for high  $\mu$  the coupling constant  $g$  (evaluated at the fixed scale  $\mu$ ) is  $\ll 1$ , we have:

$$\Delta \ll m_{gluon} \ll \mu . \quad (1.1)$$

Massless excitations dominate physical processes at very low energy with respect to the gap ( $\Delta$ ) energy. Their spectrum is intimately related to the underlying global symmetries and the way they are realized at low energies. Indeed when the dynamics is such that a continuous global symmetry is spontaneously broken a Goldstone boson appears in order to compensate for the breaking. Massless excitations obey low energy theorems governing their interactions which can be usefully encoded in effective Lagrangians. A well known example, in the regime of cold and non dense QCD, is the effective Lagrangian for pions and kaons. These Lagrangians are seen to describe well the QCD low energy phenomenology [6].

Another set of relevant constraints is provided by quantum anomalies. At zero density and temperature, 't Hooft [7] argued, using a beautiful mathematical construction, that the underlying continuous global anomalies have to be matched in a given low energy phase by a set of massless fermions associated with the intact global symmetries and a set of massless Goldstone bosons associated with the broken ones. The low energy fermions (composite or elementary) contribute via triangle diagrams while for Goldstones a Wess-Zumino term should be added to correctly implement the associated global anomalies. In this note we investigate the 't Hooft constraints for QCD at finite density.

In the next section we show, by reviewing the dynamically favored phases for  $N_f = 2, 3$  at high density that the low energy spectrum displays the correct quantum numbers to saturate the 't Hooft global anomalies.

We also observe that QCD at finite density can be envisioned, from a global symmetry and anomaly point of view, as a chiral gauge theory [8,9] for which at least part of the matter field content is in complex representations of the gauge group. Indeed an important distinction from, zero density, vector-like theories is that these theories, when strongly coupled, can exist in the Higgs phase by dynamically breaking their own gauge symmetries [8,9]. This is also the striking feature of the superconductive phase allowed for QCD at high density. In fact at finite density vector like symmetries are no longer protected against spontaneous breaking by the Vafa-Witten theorem [10].

As for a chiral gauge and in general for any gauge theory we expect the low energy massless spectrum of a finite density phase (fermions associated to intact chiral global anomalies and Goldstones for spontaneously broken symmetries) to possess the quantum numbers required by the 't Hooft anomaly matching conditions. This further global constraint, when appropriately taken into account, can help selecting the low energy phase at finite density.

## II. ANOMALY MATCHING AT FINITE DENSITY

We now show that the recently discussed superconductive QCD phases at high density do respect 't Hooft anomaly matching conditions. Specifically the spectrum of the light excitations possesses the correct quantum numbers needed to satisfy global anomaly constraints. The underlying gauge group is  $SU(3)$  while the quantum flavor group is

$$SU_L(N_f) \times SU_R(N_f) \times U_V(1) , \quad (2.1)$$

and the classical  $U_A(1)$  symmetry is destroyed at the quantum level by the Adler-Bell-Jackiw anomaly. We indicate with  $q_{\alpha;c,i}$  the two component left spinor where  $\alpha = 1, 2$  is the spin index,  $c = 1, \dots, 3$  is the color index while  $i = 1, \dots, N_f$  represents the flavor.  $\tilde{q}^{\alpha;c,j}$  is the two component conjugated right spinor. We summarize the transformation properties in the following table.

$$\begin{array}{ccccc}
[SU(3)] & SU_L(N_f) & SU_R(N_f) & U_V(1) & \\
q & \square & \square & 1 & 1 \\
\tilde{q} & \overline{\square} & 1 & \overline{\square} & -1
\end{array} . \quad (2.2)$$

The theory is subject to the following global anomalies:

$$SU_{L/R}(N_f)^3 , \quad SU_{L/R}(N_f)^2 \times U_V(1) . \quad (2.3)$$

For a vector like theory there are no further global anomalies. The cubic anomaly factor, for fermions in fundamental representations, is 1 for  $q$  and  $-1$  for  $\tilde{q}$  while the quadratic anomaly factor is always 1 leading to

$$SU_{L/R}(N_f)^3 \propto \pm 3, \quad SU_{L/R}(N_f)^2 U_V(1) \propto \pm 3. \quad (2.4)$$

We first consider the case  $N_f = 2$  which has only the  $SU_{L/R}(2)^2 \times U_V(1)$  anomaly. At zero density we have two possible phases compatible with the anomaly conditions. The first is the ordinary Goldstone phase associated with the spontaneous breaking of the underlying global symmetry to  $SU_V(2) \times U_V(1)$ . The other is the Wigner-Weyl phase where, assuming confinement, the global symmetry at low energy is intact and the needed massless spectrum consists of massless baryons with the following quantum numbers:

$$\begin{array}{ccccccc} [SU(3)] & SU_L(2) & SU_R(2) & U_V(1) & & & \\ B & 1 & \square & 1 & 3 & & \\ \tilde{B} & 1 & 1 & \overline{\square} & -3 & & \end{array} \quad (2.5)$$

Here the two component baryon emerges as composite field of the form

$$B_i = \epsilon^{abc} \epsilon^{jk} q_{a;i} q_{b;j} q_{c;k}, \quad (2.6)$$

where  $a, b, c$  and  $i, j, k$  are respectively color and flavor indices and we have omitted spin indices. A similar expression holds for  $\tilde{B}$ . The Goldstone phase is the one observed in nature. This fact supports a new idea presented in Ref. [9]. Here it is suggested that, for an asymptotically free theory<sup>†</sup>, among multiple infrared phases allowed by 't Hooft anomaly conditions the one which minimizes the entropy at the approach to freeze-out is preferred. The entropy  $S(T)$  near freeze out is given by  $S(T) = (2\pi^2/45)T^3 f_{IR}$  [9,11,12] plus higher order terms in the low temperature expansion. For the special case of an infrared-free theory,  $f_{IR}$  is simply the number of massless bosons plus 7/4 times the number of 2-component massless Weyl fermions. So the minimum entropy guide at the freeze out can be viewed via  $f_{IR}$  as a minimum degree of freedom count. In the real world case of QCD with two flavors the three Goldstone bosons lead to  $f_{IR} = 3$ , and the two massless composite fermions lead

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<sup>†</sup>We consider theories without flat directions. The possibility to extend our criterion to these theories is presently under investigation.

to  $f_{IR} = 7$ . Indeed, in this case, nature chooses to minimize  $S(T)$  at the approach to freeze out<sup>‡</sup>.

This guide, although at a very speculative level, has been used in [9] to select the infrared phase of chiral gauge theories.

What happens to the 't Hooft anomaly conditions when we squeeze nuclear matter? At very low baryon density compared to a fixed intrinsic scale of the theory  $\Lambda$ , it is reasonable to expect that the Goldstone phase persists. Hence 't Hooft anomaly conditions are still satisfied. On the other hand at very large densities it is seen, via dynamical calculations [1,2], that the ordinary Goldstone phase is no longer favored compared with a superconductive one associated with the following type of quark condensate:

$$\epsilon^{\alpha\beta}\epsilon^{abc}\epsilon^{ij} < q_{\alpha;b,i}q_{\beta;c,j} > , \quad (2.7)$$

likewise for the tilded quarks. In the following we set  $a = 3$ . This condensate is not allowed at zero density by the Vafa-Witten theorem. One can ask if the spectrum of low energy excitations still possesses the correct quantum numbers to satisfy 't Hooft anomaly conditions. The previous condensate breaks the gauge symmetry while leaving intact the following group:

$$[SU(2)] \times SU_L(2) \times SU_R(2) \times \tilde{U}_V(1) , \quad (2.8)$$

where  $[SU(2)]$  is the unbroken part of the gauge group. The  $\tilde{U}_V(1)$  generator is the following linear combination of the previous  $U_V(1)$  generator  $Q_V$  and the broken diagonal generator of the  $SU(3)$  gauge group  $Q_8 = \text{diag}\{1, 1, -2\}$

$$\tilde{Q} = Q_V - Q_8 , \quad (2.9)$$

Then the  $\tilde{Q}$  charge of the quarks with color 1 and 2 is zero.

The low energy massless excitations are the quarks not participating in the condensate. We summarize their symmetry properties as follows:

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<sup>‡</sup>Of course, at any finite  $T$ , the preferred phase is chosen from among all the states entering the partition function by minimizing the free energy density, which becomes the energy density at  $T = 0$ . Comparing these quantities for different states when the theory is strongly interacting is, however, generally a strong coupling problem.

$$[SU(2)] \ SU_L(2) \ SU_R(2) \ \tilde{U}_V(1)$$

$$\begin{array}{ccccccc} q_{\alpha;3,i} & 1 & \square & 1 & 3 & , & (2.10) \\ \tilde{q}^{\alpha;3,j} & 1 & 1 & \overline{\square} & -3 & & \end{array}$$

These massless low energy fermions correctly match the 't Hooft anomaly conditions. In Fig. 1 we draw the relevant diagrams for the global anomalies. At low energies we use the elementary massless fields  $q_{3,i}$  ( $\tilde{q}^{3,i}$ ) with non zero  $\tilde{Q}$  charge contributing to the first diagram. At high energies we should consider the underlying symmetries and hence we split the left hand side triangle into the two right hand side diagrams and we exchange all of the underlying fermions. Since the second diagram contains only one gauge generator it vanishes identically and the 't Hooft anomaly conditions are matched.

Our proof is part of a more general theorem established for chiral gauge theories at zero density. The theorem states that if in a theory of massless fermions, one breaks the gauge symmetry in such a way that a subset of the original fermions remain massless, those massless fermions always obey 't Hooft anomaly conditions with respect to the unbroken chiral symmetries [13].

The superconductive phase for  $N_f = 2$  possesses the same global symmetry group of the confined Wigner-Weyl phase. This remarkable feature when considering chiral gauge theories at zero density (where the superconductive phase is now a Higgs phase) is referred as complementarity. This idea was introduced in Ref. [14] where it was conjectured that any Higgs phase can be described in terms of confined degrees of freedom and vice versa. We stress that complementarity does not imply that an Higgs phase and a confined phase are physically equivalent. Indeed the elementary or composite nature of the low energy particles can be uncovered via scattering experiments. Hence, finite density QCD, at least from the global symmetry and anomalies point of view, resembles a chiral gauge theory.

However since the minimum entropy guide has been stated only at zero density it should not be used to help selecting a phase at high density. Clearly at low densities, where we expect the theory to behave like ordinary QCD, the conjecture holds.

Let us consider now the case of  $N_f = 3$  light flavors. At zero density only the Goldstone phase is allowed and the resulting symmetry group is  $SU_V(3) \times U_V(1)$ . Indeed there is no solution of the 't Hooft anomaly condition with massless composite fermions leaving intact the flavor group. In this case the topological Wess-Zumino term for the Goldstone bosons can be constructed to correctly implement the global anomalies of the underlying theory at

$$\tilde{Q} \text{ (wavy)} \triangle_{SU_{L/R}(2)} = Q_V \text{ (wavy)} \triangle_{SU_{L/R}(2)} - Q_8 \text{ (wavy)} \triangle_{SU_{L/R}(2)}$$

FIG. 1.

the effective Lagrangian level. The Vafa-Witten theorem for vector-like theories prohibits the breaking of vector like symmetries like  $U_V(1)$ .

Turning on low baryon density we expect to remain in the confined phase with the same number of Goldstone bosons (i.e. 8). Evidently the 't Hooft anomaly conditions are satisfied. At very high density, dynamical computations suggest [3] that the preferred phase is a superconductive one and the following ansatz for a quark-quark type of condensate is energetically favored:

$$\epsilon^{\alpha\beta} \langle q_{\alpha;a,i} q_{\beta;b,j} \rangle \sim k_1 \delta_{ai} \delta_{bj} + k_2 \delta_{aj} \delta_{bi} , \quad (2.11)$$

where we have a similar expression for the tilded fields. The condensate breaks completely the gauge group while locking together the left/right transformations to color. The final global symmetry group is

$$SU_{c+L+R}(3) , \quad (2.12)$$

and the low energy spectrum consists of 9 Goldstone bosons. The effective Lagrangian at low energies [15] for the Goldstones is similar to the ordinary effective Lagrangian for QCD at zero density except for an extra Goldstone boson associated with the spontaneously broken  $U_V(1)$  symmetry. In this case we expect the underlying global anomalies to be matched at low energies via the Wess-Zumino term. It is instructive to explicitly construct this term. This term is also been discussed together with its relation with flavor anomalies at finite

density in Ref. [16]. However in this paper, the two flavors QCD case is not investigated and the possibility to envision QCD at finite density as a chiral gauge theory is not discussed.

The Goldstone bosons are encoded in the unitary matrix  $U$  transforming linearly under the left-right flavor rotations

$$U \rightarrow g_L U g_R^\dagger . \quad (2.13)$$

with  $g_{L/R} \in SU_{L/R}(N_f)$ . In our notation  $U$  is the transpose of  $\Sigma$  defined in Ref. [15].  $U$  satisfies the non linear realization constraint  $UU^\dagger = 1$ . We also require  $\det U = 1$ . In this way we avoid discussing the axial  $U_A(1)$  anomaly at the effective lagrangian level. (see Ref. [17] for a general discussion of trace and  $U_A(1)$  anomaly). We have

$$U = e^{i\frac{\Phi}{F^2}} , \quad (2.14)$$

with  $\Phi = \sqrt{2}\Phi^a T^a$  representing the 8 Goldstone bosons.  $T^a$  are the generators of  $SU(3)$ , with  $a = 1, \dots, 8$  and  $\text{Tr} [T^a T^b] = \frac{1}{2}\delta^{ab}$ .  $F$  is the Goldstone bosons decay constant at finite density.

The effective Lagrangian globally invariant under chiral rotations is (up to two derivatives and counting  $U$  as a dimensionless field)

$$L = \frac{F^2}{2} \text{Tr} [\partial_\mu U \partial^\mu U^\dagger] . \quad (2.15)$$

The Wess-Zumino term [18] can be compactly written using the language of differential forms. It is useful to introduce the algebra valued Maurer-Cartan one form:

$$\alpha = (\partial_\mu U) U^{-1} dx^\mu \equiv (dU) U^{-1} . \quad (2.16)$$

The Wess-Zumino effective action is

$$\Gamma_{WZ} [U] = C \int_{M^5} \text{Tr} [\alpha^5] . \quad (2.17)$$

The price to pay in order to make the action local is to augment by one the space dimensions. Hence the integral must be performed over a five-dimensional manifold whose boundary ( $M^4$ ) is the ordinary Minkowski space. The constant  $C$ , at zero density, is fixed to be

$$C = -i \frac{N}{240\pi^2} , \quad (2.18)$$

by comparing the current algebra prediction for the time honored process  $\pi^0 \rightarrow 2\gamma$  with the amplitude predicted using Eq. (2.17) once we gauge the electromagnetic sector [19,20] of the Wess-Zumino term, and  $N$  is the number of colors (fixed to be 3 in this case). To



the previous lagrangian one can still add the extra Goldstone boson associated to the  $U_V(1)$  symmetry without altering the previous discussion (see [15]). One can check that the global anomalies are correctly implemented by carefully gauging the Wess-Zumino term [19,20] with respect to the flavor symmetries. Hence for the 3 flavor case too, the 't Hooft global anomalies are matched at finite (low and high) density.

In writing the Goldstone Lagrangian we have not yet considered the breaking of Lorentz invariance at finite density. Following Ref. [15] we note that the Goldstones obey, in medium, a linear dispersion relation of the type  $E = v|\vec{p}|$ , where  $E$  and  $|\vec{p}|$  are respectively the energy and the momentum of the Goldstone bosons. By rescaling the vector coordinates  $\vec{x} \rightarrow \vec{x}/v$  the Lagrangian in Eq. (2.15) becomes:

$$L = \frac{F^2}{2} \text{Tr} \left[ \dot{U} \dot{U}^\dagger - v^2 \vec{\nabla} U \cdot \vec{\nabla} U^\dagger \right] . \quad (2.19)$$

$\alpha$ , being a differential form, is unaffected by a coordinate rescaling leaving unaltered the form of the Wess-Zumino term.

Due to the breaking of the baryon number the final global symmetry group in the superconductive phase differs from the ordinary Goldstone phase. Now we cannot regard this phase as complementary to the confined one. As the baryon density decreases we also expect the spectrum of light Goldstone bosons to change abruptly at the phase transition.

Recently in Ref. [21] QCD for  $N_f$  larger than 3 has been investigated at high density. All the phases discovered seem to involve the breaking of global symmetries to a subgroup of the vector like subgroup. These phases automatically respect 't Hooft anomaly conditions.

### III. CONCLUSIONS

We noted, by reviewing the dynamically favored phases for  $N_f = 2, 3$  at high density that the low energy spectrum possesses the correct quantum numbers to saturate the 't Hooft anomaly matching conditions.

We have also argued that QCD at finite density can be thought, from the point of view of global symmetry and anomalies, as a chiral gauge theory [8]. Indeed at finite density the vector like symmetries are no longer protected against spontaneous breaking by the Vafa-Witten theorem. As for a chiral gauge and in general for any gauge theory we expect the low energy massless spectrum of a finite density phase (fermions associated to intact chiral global anomalies and Goldstones for spontaneously broken symmetries) to possess the quantum numbers required by the 't Hooft anomaly matching conditions. This further

global constraint should be appropriately taken into account when selecting the low energy finite density phase.

We stress that, while dynamical calculations rely on the high-density approximation, t'Hooft anomaly matching conditions are global constraints and hence applicable to low as well as high densities.

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